

Some resolvent set properties of band operators with matrix elements.

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Abstract

For operators generated by a certain class of infinite band matrices with matrix elements we establish a characterization of the resolvent set in terms of polynomial solutions of the underlying higher order finite difference equations. This enables us to describe some asymptotic behaviour of the corresponding systems of vector orthogonal polynomials on the resolvent set.

Keywords and phrases. Band matrices, Difference operators, Weyl matrix, Orthogonal polynomials.

1 Introduction

In recent years some results concerning the structure of the resolvent set of nonsymmetric difference operators generated by infinite band matrices were obtained [1]- [5]. Originally the second order difference operators generated by three-diagonal matrices were studied [1] - [3], later on some results on the resolvent set properties of higher order band operators were obtained [4] - [5]. In studying the spectral properties of such band operators the important role is played by the Weyl matrix [5] (or the Weyl function in the second order case [1]- [2]) of the corresponding operator, the systems of polynomials orthogonal with respect to the Weyl functions as well as the convergence of the Hermite-Pade approximations of the Weyl matrix. Here we obtain some results similar to [5] in the case of higher order band operators with matrix elements. Also note that for the second order band operators with matrix elements a similar results on the structure of their resolvent sets were obtained in [6].

Consider an infinite nonsymmetric band matrix $A = (A_{k,l})_{k,l=0}^{\infty}$ whose entries are matrices of order N with complex elements: $A_{k,l} \in \mathbb{C}^{N \times N}$, satisfying for all k and for all $\ell < k - s$ or $\ell > k + r$

$$A_{k,\ell} = O, \quad A_{k,k+r}, A_{k+s,k} \quad \text{are invertible,} \quad (1)$$

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where O is a zero matrix of order N . That is, A takes the form

$$A = \begin{pmatrix} A_{0,0} & \dots & A_{0,r} & O & O & \dots & \dots & \dots \\ A_{1,0} & A_{1,1} & \dots & A_{1,r+1} & O & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & \\ A_{s,0} & A_{s,1} & A_{s,2} & \dots & \dots & A_{s,s+r} & O & \dots \\ O & A_{s+1,1} & A_{s+1,2} & A_{s+1,3} & \dots & \dots & A_{s+1,s+r+1} & \dots \\ O & O & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $s + r + 1$ nontrivial diagonals, where r and s are some fixed natural numbers. To the matrix A we assign finite-difference equations in the matrices $Y_k, Y_k^+ \in \mathbb{C}^{N \times N}$

$$A_{k,k-s}Y_{k-s} + A_{k,k-s+1}Y_{k-s+1} + \dots + A_{k,k+r}Y_{k+r} = \lambda Y_k, \quad (2)$$

$$Y_{k-r}^+ A_{k-r,k} + Y_{k-r+1}^+ A_{k-r+1,k} + \dots + Y_{k+s}^+ A_{k+s,k} = \lambda Y_k^+, \quad (3)$$

where $k \geq 0$, $\lambda \in \mathbb{C}$ is some parameter, and we define $A_{k,\ell}$ with negative indices as

$$\begin{aligned} A_{k,k-s} &= -E, \quad A_{k-k-s+j,k} = O, \quad 0 \leq k < s, \quad 1 \leq j < s-k; \\ A_{k-r,k} &= -E, \quad A_{k-r+j,k} = O, \quad 0 \leq k < r, \quad 1 \leq j < r-k; \end{aligned}$$

where E is a unit matrix.

We consider some particular fundamental systems of solutions $\{P(\lambda), Q(\lambda)\}$ of (2), and $\{P^+(\lambda), Q^+(\lambda)\}$ of (3), respectively, with elements being polynomials with respect to λ with matrix coefficients: Denote by

$$\begin{aligned} Q(\lambda) &= (Q_k(\lambda))_{k=-s}^\infty = (Q_k^1(\lambda), \dots, Q_k^r(\lambda))_{k=-s}^\infty; \\ P(\lambda) &= (P_k(\lambda))_{k=-s}^\infty = (P_k^1(\lambda), \dots, P_k^s(\lambda))_{k=-s}^\infty; \end{aligned}$$

solutions of (2) satisfying the initial conditions

$$Q_{0:r-1} = I_r, \quad Q_{-s:-1} = \mathbf{0}_{s \times r}, \quad P_{-s:-1} = I_s, \quad P_{0:r-1} = \mathbf{0}_{r \times s}. \quad (4)$$

Furthermore, denote by

$$Q^+(\lambda) = \left(Q_k^{1,+}(\lambda) \right)_{k=-r}^\infty, \quad P^+(\lambda) = \left(P_k^{1,+}(\lambda) \right)_{k=-r}^\infty$$

solutions of the dual recurrence relation (3) satisfying the initial conditions

$$Q_{0:s-1}^+ = I_s, \quad Q_{-r:-1}^+ = \mathbf{0}_{s \times r}, \quad P_{-r:-1}^+ = I_r, \quad P_{0:s-1}^+ = \mathbf{0}_{r \times s}. \quad (5)$$

Here and in what follows we use the following notations: $Q_{j:k}$ (and $Q_{j:k}^+$, respectively) for the stacked matrix with rows Q_ℓ , $\ell = j, j+1, \dots, k$ (with columns Q_ℓ^+ , $\ell = j, j+1, \dots, k$), etc; $I_i, \mathbf{0}_{i \times j}$ for the identity and zero matrices of sizes i and $i \times j$ (with the elements from $\mathbb{C}^{N \times N}$, $I_1 = E$). We shall also use $A_{j:k,m:n}$ for the submatrix of A composed of its rows labeled j to k , and its columns labeled m to n . Finally, we use block matrix notations like $[M \ N]$.

Lemma 1. *The expression*

$$F(k) \equiv \sum_{i=0}^{s-1} \sum_{j=i+1}^s Y_{k+i}^+ A_{k+i,k+i-j} Y_{k+i-j} - \sum_{i=0}^{r-1} \sum_{j=i+1}^r Y_{k+i-j}^+ A_{k+i-j,k+i} Y_{k+i}, \quad k \geq 0$$

does not depend on k

Proof. We have

$$\begin{aligned} F(k) &= Y_k^+ A_{k,k-1} Y_{k-1} + Y_k^+ A_{k,k-2} Y_{k-2} + \cdots + Y_k^+ A_{k,k-s} Y_{k-s} + \\ &+ Y_{k+1}^+ A_{k+1,k-1} Y_{k-1} + Y_{k+1}^+ A_{k+1,k-2} Y_{k-2} + \cdots + Y_{k+1}^+ A_{k+1,k+1-s} Y_{k+1-s} + \\ &+ \cdots + Y_{k+s-1}^+ A_{k+s-1,k-1} Y_{k-1} - Y_{k-1}^+ A_{k-1,k} Y_k - Y_{k-2}^+ A_{k-2,k} Y_k - \cdots - Y_{k-r}^+ A_{k-r,k} Y_k - \\ &- Y_{k-1}^+ A_{k-1,k+1} Y_{k+1} - Y_{k-2}^+ A_{k-2,k+1} Y_{k+1} - \cdots - Y_{k-r+1}^+ A_{k-r+1,k} Y_{k+1} - \\ &- \cdots - Y_{k-1}^+ A_{k-1,k+r-1} Y_{k+r-1}. \end{aligned}$$

Applying (2)- (3) to $F(k)$ for $i = 0$, we get

$$\begin{aligned} F(k) &= Y_k^+ (\lambda - A_{k,k}) Y_k - Y_k^+ A_{k,k+1} Y_{k+1} - \cdots - Y_k^+ A_{k,k+r} Y_{k+r} + \\ &+ Y_{k+1}^+ A_{k+1,k-1} Y_{k-1} + Y_{k+1}^+ A_{k+1,k-2} Y_{k-2} + \cdots + Y_{k+1}^+ A_{k+1,k+1-s} Y_{k+1-s} + \\ &+ \cdots + Y_{k+s-1}^+ A_{k+s-1,k-1} Y_{k-1} + Y_k^+ (A_{k,k} - \lambda) Y_k + Y_{k+1}^+ A_{k+1,k} Y_k + \cdots + Y_{k+s}^+ A_{k+s,k} Y_k - \\ &- Y_{k-1}^+ A_{k-1,k+1} Y_{k+1} - Y_{k-2}^+ A_{k-2,k+1} Y_{k+1} - \cdots - Y_{k-r+1}^+ A_{k-r+1,k} Y_{k+1} - \\ &- \cdots - Y_{k-1}^+ A_{k-1,k+r-1} Y_{k+r-1}. \end{aligned}$$

By separating the “positive” and “negative” parts of $F(k)$, we obtain

$$\begin{aligned} F(k) &= -Y_k^+ A_{k,k+2} Y_{k+2} - \cdots - Y_k^+ A_{k,k+r} Y_{k+r} + \\ &+ Y_{k+1}^+ A_{k+1,k-1} Y_{k-1} + Y_{k+1}^+ A_{k+1,k-2} Y_{k-2} + \cdots + Y_{k+1}^+ A_{k+1,k+1-s} Y_{k+1-s} + \\ &+ \cdots + Y_{k+s-1}^+ A_{k+s-1,k-1} Y_{k-1} + \\ &+ Y_{k+2}^+ A_{k+2,k} Y_k + \cdots + Y_{k+s}^+ A_{k+s,k} Y_k - \\ &- Y_k^+ A_{k,k+1} Y_{k+1} - Y_{k-1}^+ A_{k-1,k+1} Y_{k+1} - \cdots - Y_{k-r+1}^+ A_{k-r+1,k} Y_{k+1} - \\ &- \cdots - Y_{k-1}^+ A_{k-1,k+r-1} Y_{k+r-1} = \cdots \\ &= Y_{k+1}^+ A_{k+1,k} Y_k + Y_{k+1}^+ A_{k+1,k-1} Y_{k-1} + \cdots + Y_{k+1}^+ A_{k+1,k+1-s} Y_{k+1-s} + \\ &+ Y_{k+2}^+ A_{k+2,k} Y_k + Y_{k+2}^+ A_{k+2,k-1} Y_{k-1} + \cdots + Y_{k+2}^+ A_{k+2,k+2-s} Y_{k+2-s} + \\ &+ \cdots + Y_{k+s}^+ A_{k+s,k} Y_k - Y_k^+ A_{k,k+1} Y_{k+1} - Y_{k-1}^+ A_{k-1,k+1} Y_{k+1} - \cdots - Y_{k-r+1}^+ A_{k-r+1,k+1} Y_{k+1} - \\ &- Y_k^+ A_{k,k+2} Y_{k+2} - Y_{k-1}^+ A_{k-1,k+2} Y_{k+2} - \cdots - Y_{k-r+2}^+ A_{k-r+2,k+1} Y_{k+2} - \\ &- \cdots - Y_k^+ A_{k,k+r} Y_{k+r} = F(k+1). \end{aligned}$$

□

Lemma 2. *For all $k \geq 0$*

$$I_{r+s} = \begin{bmatrix} P_{k-r:k+s-1}^+ \\ Q_{k-r:k+s-1}^+ \end{bmatrix} \times \quad (6)$$

$$\times \begin{bmatrix} \mathbf{0}_{\mathbf{r} \times \mathbf{s}} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & \mathbf{0}_{\mathbf{s} \times \mathbf{r}} \end{bmatrix} \times [Q_{k-s:k+r-1}, -P_{k-s:k+r-1}] \quad (7)$$

Proof. For $k = 0$, the claim follows from the choice (4) and (5) of the initial conditions and the definition of $A_{k,\ell}$ with negative indices. Now consider the particular solutions $Y_n = Q_n^m$ and $Y_n^+ = P_n^{\ell,+}$ for some indices $\ell, m = 1, \dots, r$. We find, by using (1) that

$$\begin{aligned} G^{\ell,m}(k) &\equiv \sum_{i=0}^{s-1} \sum_{j=1}^s P_{k+i}^{\ell,+} A_{k+i,k+i-j} Q_{k+i-j}^m - \sum_{i=0}^{r-1} \sum_{j=1}^r P_{k+i-j}^{\ell,+} A_{k+i-j,k+i} Q_{k+i}^m = \\ &= \sum_{i=0}^{s-1} \sum_{j=i+1}^s P_{k+i}^{\ell,+} A_{k+i,k+i-j} Q_{k+i-j}^m - \sum_{i=0}^{r-1} \sum_{j=i+1}^r P_{k+i-j}^{\ell,+} A_{k+i-j,k+i} Q_{k+i}^m \end{aligned}$$

According to Lemma 1, for $k > 0$, $G^{\ell,m}(k) = G^{\ell,m}(0) = \delta_{\ell,m}$. In other words, we have shown that the corresponding entries in the first r rows and columns of the claimed matrix identity coincide. The identities for the other three blocks : $r \times s$, $s \times r$ and $s \times s$ are obtained in a similar way by choosing $Y_n \in \{P_n^m, Q_n^m\}$ and $Y_n^+ \in \{P_n^{\ell,+}, Q_n^{\ell,+}\}$. \square

2 Band operators and their resolvent set properties

The above matrix A generates a linear operator in the space l_N^2 of sequences $u = (u_0, u_1, \dots)$ where the vector column $u_j \in \mathbb{C}^N$, with inner product $(u, v) = \sum_{j=0}^{\infty} v_j^* u_j$. For this operator we shall use the same notation.

Let I be the identity operator in l_N^2 . Then it admits the matrix representation

$$I = \begin{pmatrix} E & O & O & \dots \\ O & E & O & \dots \\ O & O & E & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let $\mathfrak{M} = (\mathfrak{M}_{i,j})_{i=1,\dots,r}^{j=1,\dots,s}$, $\mathfrak{M}_{i,j} \in \mathbb{C}^{N \times N}$ be an arbitrary matrix of the size $r \times s$ with matrix elements. Then for $k, n \in \mathbb{Z}_+$ we define

$$R_{k,n} = \begin{cases} Q_k(\lambda) R_n^+(\lambda), & 0 \leq k < n + r, \\ R_k(\lambda) Q_n^+(\lambda), & 0 \leq n < k + s \end{cases} \quad (8)$$

where $R_k(\lambda) = Q_k(\lambda) \mathfrak{M} - P_k(\lambda)$ and $R_n^+(\lambda) = \mathfrak{M} Q_n^+(\lambda) - P_n^+(\lambda)$.

As we see, we have two different definitions of $R_{k,n}$ for $n - s < k < n + r$. In fact, they give the same value.

Lemma 3. For $n - s < k < n + r$ $Q_k(\lambda) R_n^+(\lambda) = R_k(\lambda) Q_n^+(\lambda)$.

Proof. As follows from the definition of $R_k(\lambda)$, $R_n^+(\lambda)$, it suffices to show that

$$Q_k(\lambda) P_n^+(\lambda) = P_k(\lambda) Q_n^+(\lambda), \quad n - s < k < n + r. \quad (9)$$

Consider the case $s = r = 1$. By Lemma 2 we have

$$\begin{cases} P_k^+(\lambda) A_{k,k-1} Q_{k-1}(\lambda) - P_{k-1}^+(\lambda) A_{k-1,k} Q_k(\lambda) = E \\ P_k^+(\lambda) A_{k,k-1} P_{k-1}(\lambda) - P_{k-1}^+(\lambda) A_{k-1,k} P_k(\lambda) = O \\ Q_k^+(\lambda) A_{k,k-1} Q_{k-1}(\lambda) - Q_{k-1}^+(\lambda) A_{k-1,k} Q_k(\lambda) = O \\ -Q_k^+(\lambda) A_{k,k-1} P_{k-1}(\lambda) + Q_{k-1}^+(\lambda) A_{k-1,k} P_k(\lambda) = E. \end{cases} \quad (10)$$

Now for $k \geq 0$ consider the system

$$\begin{cases} Q_{k-1}(\lambda)\Delta_k^1 + P_{k-1}(\lambda)\Delta_k^2 = O \\ A_{k-1,k}Q_k(\lambda)\Delta_k^1 + A_{k-1,k}P_k(\lambda)\Delta_k^2 = E, \end{cases}$$

where Δ_k^1, Δ_k^2 are unknown. Multiplying the first equation of the system on the left by $P_k^+(\lambda)A_{k,k-1}$ and the second equation by $-P_{k-1}^+(\lambda)$ and summing the resulting equations, we obtain

$$\begin{aligned} & (P_k^+(\lambda)A_{k,k-1}Q_{k-1}(\lambda) - P_{k-1}^+(\lambda)A_{k-1,k}Q_k(\lambda))\Delta_k^1 + \\ & + (P_k^+(\lambda)A_{k,k-1}P_{k-1}(\lambda) - P_{k-1}^+(\lambda)A_{k-1,k}P_k(\lambda))\Delta_k^2 = -P_{k-1}^+(\lambda). \end{aligned}$$

Applying (10) we find $\Delta_k^1 = -P_{k-1}^+(\lambda)$.

Similarly, multiplying the first equation of the system on the left by $-Q_k^+(\lambda)A_{k,k-1}$ and the second equation by $Q_{k-1}(\lambda)$ we find $\Delta_k^2 = Q_{k-1}^+(\lambda)$.

Thus $Q_n(\lambda)P_n^+(\lambda) = P_n(\lambda)Q_n^+(\lambda)$ $n \geq 0$ and therefore $Q_n(\lambda)R_n^+(\lambda) = R_n(\lambda)Q_n^+(\lambda)$.

For an arbitrary s and r the proof is similar as above (and based on Lemma 2). \square

Now consider the infinite matrix $R = (R_{k,n})_{k,n=0}^\infty$

Lemma 4. *The following matrix identities (formal products between infinite matrices)*

$$(\lambda I - A)R = I, \quad R(\lambda I - A) = I \quad (11)$$

are hold.

Proof. By symmetry (replace A by its transposed), it is sufficient to show the first identity of (11). Since $(Q_n(\lambda))_{n \geq 0}$ is a solution of (2), we have

$$(\lambda I - A) \begin{bmatrix} R_{0:k-1,n} \\ \mathbf{0}_{\infty \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(k-r) \times 1} \\ A_{k-r:k-1,k:k+r-1}R_{k:k+r-1,n} \\ -A_{k:k+s-1,k-s:k-1}R_{k-s:k-1,n} \\ \mathbf{0}_{\infty \times 1} \end{bmatrix}, \quad r \leq k \leq n. \quad (12)$$

Similarly, $(R_n(\lambda))_{n \geq 0}$ is a solution of (2), and hence we have the (formal) identity

$$(\lambda I - A) \begin{bmatrix} \mathbf{0}_{n \times 1} \\ R_{n:\infty,n} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(n-r) \times 1} \\ -A_{n-r:n-1,n:n+r-1}R_{n:n+r-1,n} \\ A_{n:n+s-1,n-s:n-1}R_{n-s:n-1}(\lambda)Q_n^+(\lambda) \\ \mathbf{0}_{\infty \times 1} \end{bmatrix}. \quad (13)$$

Combining identity (12) for $k = n$ with (13) leads to the (formal) identity

$$(\lambda I - A)R_{0:\infty,n} = \begin{bmatrix} \mathbf{0}_{n \times 1} \\ A_{n:n+s-1,n-s:n-1}[R_{n-s:n-1}(\lambda)Q_n^+(\lambda) - R_{n-s:n-1,n}] \\ \mathbf{0}_{\infty \times 1} \end{bmatrix}$$

Using the definition of $R_{k,n}$ and (9) we have

$$A_{n:n+s-1,n-s:n-1}[R_{n-s:n-1}(\lambda)Q_n^+(\lambda) - R_{n-s:n-1,n}] = \begin{bmatrix} A_{n,n-s}(Q_{n-s}(\lambda)P_n^+(\lambda) - P_{n-s}(\lambda)Q_n^+(\lambda)) \\ \mathbf{0}_{(s-1) \times 1} \end{bmatrix}$$

It remains to show that

$$A_{n,n-s}(-P_{n-s}(\lambda)Q_n^+(\lambda) + Q_{n-s}(\lambda)P_n^+(\lambda)) = E.$$

In doing so, we use the first row of the matrix identity (6) and (9).

For example, for $s = 1$ and $r = 2$ the first row of (6) gives

$$\begin{aligned} P_n^{1,+}A_{n,n-1}Q_{n-1}^1 - (P_{n-2}^{1,+}A_{n-2,n} - P_{n-1}^{1,+}A_{n-1,n})Q_n^1 - P_{n-1}^{1,+}A_{n-1,n+1}Q_{n+1}^1 &= E \\ P_n^{1,+}A_{n,n-1}Q_{n-1}^2 - (P_{n-2}^{1,+}A_{n-2,n} - P_{n-1}^{1,+}A_{n-1,n})Q_n^2 - P_{n-1}^{1,+}A_{n-1,n+1}Q_{n+1}^2 &= O \\ P_n^{1,+}A_{n,n-1}P_{n-1} - (P_{n-2}^{1,+}A_{n-2,n} - P_{n-1}^{1,+}A_{n-1,n})P_n - P_{n-1}^{1,+}A_{n-1,n+1}P_{n+1} &= O. \end{aligned} \quad (14)$$

At the same time, from (9) it follows that

$$\begin{aligned} Q_n^1P_n^{+,1} + Q_n^2P_n^{+,2} &= P_nQ_n^+ \\ Q_{n+1}^1P_n^{+,1} + Q_{n+1}^2P_n^{+,2} &= P_{n+1}Q_n^+ \end{aligned} \quad (15)$$

Multiplying the first equation of (14) on the right by $P_n^{1,+}$, the second equation by $P_n^{2,+}$ and the third, respectively, by $-Q_n^+$ and summing the resulting equations we obtain, using (15) that

$$P_n^{1,+}A_{n,n-1}(-P_{n-1}Q_n^+ + Q_{n-1}^1P_n^{1,+} + Q_{n-1}^2P_n^{2,+}) = P_n^{1,+}$$

Thus

$$A_{n,n-1}(-P_{n-1}Q_n^+ + Q_{n-1}^1P_n^{1,+} + Q_{n-1}^2P_n^{2,+}) = A_{n,n-1}(-P_{n-1}Q_n^+ + Q_{n-1}P_n^+) = E,$$

and therefore

$$(\lambda I - A)R_{0:\infty,n} = \begin{bmatrix} \mathbf{0}_{n \times 1} \\ E \\ \mathbf{0}_{\infty \times 1} \end{bmatrix}.$$

This shows claim (11). Notice that in the above reasoning we require that $n \geq r$. A proof for the case $0 \leq n < r$ is similar, we omit the technical details. \square

Now consider the case

$$\sup_{i,j \geq 0} \|A_{i,j}\| \leq C < \infty, \quad (16)$$

where $\|\cdot\|$ is a certain matrix norm. Then the operator A is bounded. Recall that λ is an element of the resolvent set $\Omega(A)$ if there exists an operator $\mathcal{R}(\lambda) = (\lambda I - A)^{-1} \in L(l_N^2)$ referred to as the resolvent of A such that $(\lambda I - A)\mathcal{R}(\lambda)u = u$ and $\mathcal{R}(\lambda)(\lambda I - A)v = v$ for any u and $v \in l_N^2$. As the operator A , the resolvent $\mathcal{R}(\lambda)$ can be expressed as an infinite matrix with matrix elements of order N : $\mathcal{R}(\lambda) = (\tilde{R}_{i,j})_{i,j=0}^\infty$, $\tilde{R}_{i,j} \in \mathbb{C}^{N \times N}$.

The matrix

$$\mathcal{M}(\lambda, A) \equiv (\tilde{R}_{i,j})_{i=0,\dots,r-1}^{j=0,\dots,s-1}$$

is called the Weyl matrix of the operator A . Note that the properties of the Weyl matrix in a more general case of A with operator elements were studied in [7], where it was shown that $Q(\lambda)$ and $Q^+(\lambda)$ are the systems of polynomials, orthogonal with respect to $\mathcal{M}(\lambda, A)$.

For the bounded operators A we may establish the following criterion for $\Omega(A)$.

Theorem 1. Suppose that the band operator A with matrix representation (1) satisfies (16). Then $\lambda \in \mathbb{C}$ belongs to the resolvent set of A if and only if there exist positive constants C , $q < 1$ and a matrix $\mathfrak{M} = (\mathfrak{M}_{i,j})_{i=1,\dots,r}^{j=1,\dots,s}$, $\mathfrak{M}_{i,j} \in \mathbb{C}^{N \times N}$ such that

$$\|R_{k,n}\| \leq Cq^{|n-k|}, \quad k, n \in \mathbb{Z}_+, \quad (17)$$

where $R_{k,n}$ are defined by (8). In this case, the matrix $\mathfrak{M} = \mathfrak{M}(\lambda) = (R_{i,j})_{i=0,\dots,r-1}^{j=0,\dots,s-1}$ is unique, and coincides with the Weyl matrix $\mathcal{M}(\lambda, A)$.

Proof. Necessity. Let $\lambda \in \Omega(A)$. Assume that $\tilde{R} = (\tilde{R}_{i,j})_{i,j=0}^\infty$ is the matrix representation of $\mathcal{R}(\lambda)$. Take $\mathfrak{M} = (\tilde{R}_{i,j})_{i=0,\dots,r-1}^{j=0,\dots,s-1}$ and consider the matrix $R = (R_{k,n})_{k,n=0}^\infty$, where $R_{k,n}$ are defined by (8). From the resolvent identity $(\lambda I - A)\mathcal{R}(\lambda) = I$ together with Lemma 4 and (4) follows that R and \tilde{R} satisfy the same recurrence relation and the same initializations; hence they coincide. It remains to show the decay rate (17). In the scalar case ($A_{i,j} \in \mathbb{C}$) it follows from the result of [8] on the decay rate of the elements of the inverses of band matrices; for $A_{i,j} \in \mathbb{C}^{N \times N}$ it can be proved in a similar manner.

Sufficiency. Assume that the conditions of the theorem are satisfied. As above, we build up the infinite matrix $R = (R_{k,n})_{k,n=0}^\infty$. From Lemma 4 and (17) it follows that we may correctly define the operator $(A - \lambda I)^{-1}$ on the basis vectors from l_N^2 and therefore on the finite vectors (in this basis). Also, from (17) it follows that the operator $(A - \lambda I)^{-1}$ defined on the finite vectors, is bounded. Thus we can extend the $(A - \lambda I)^{-1}$ on all l_N^2 and therefore $\lambda \in \Omega(A)$. \square

Corollary 1. If $\lambda \in \Omega(A)$ and A satisfies the conditions of Theorem 1, then

$$\limsup_{k \rightarrow \infty} \|R_k^j(\lambda)\|^{\frac{1}{k}} < 1, \quad \limsup_{n \rightarrow \infty} \|R_n^{i,+}(\lambda)\|^{\frac{1}{n}} < 1, \quad i = 1, \dots, r; \quad j = 1, \dots, s. \quad (18)$$

Theorem 2. If $\lambda \in \Omega(A)$ and A satisfies the conditions of Theorem 1, then for $i = 1, \dots, r$, and $j = 1, \dots, s$ there holds

$$\limsup_{k \rightarrow \infty} \|Q_k^i(\lambda)\|^{\frac{1}{k}} > 1, \quad \limsup_{k \rightarrow \infty} \|Q_k^{j,+}(\lambda)\|^{\frac{1}{k}} > 1. \quad (19)$$

Proof. Multiplying the equation of Lemma 2 on the left and on the right with

$$\begin{bmatrix} I_r & -\mathfrak{M} \\ \mathbf{0}_{s \times r} & I_s \end{bmatrix}, \quad \begin{bmatrix} I_r & \mathfrak{M} \\ \mathbf{0}_{s \times r} & I_s \end{bmatrix}$$

gives

$$\begin{aligned} I_{r+s} &= \begin{bmatrix} -R_{k-r:k+s-1}^+ \\ Q_{k-r:k+s-1}^+ \end{bmatrix} \\ &\cdot \begin{bmatrix} \mathbf{0}_{r \times s} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & \mathbf{0}_{s \times r} \end{bmatrix} \cdot \begin{bmatrix} Q_{k-s:k+r-1} R_{k-s:k+r-1} \end{bmatrix} \end{aligned} \quad (20)$$

and hence in particular

$$Q_{k-r:k+s-1}^+ \cdot \begin{bmatrix} \mathbf{0}_{r \times s} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & \mathbf{0}_{s \times r} \end{bmatrix} \cdot R_{k-s:k+r-1} = I_s.$$

Therefore

$$\sum_{m=0}^{s-1} \sum_{n=1}^s Q_{k+m}^{j,+} A_{k+m,k+m-n} R_{k+m-n}^j - \sum_{m=0}^{r-1} \sum_{n=1}^r Q_{k+m-n}^{j,+} A_{k+m-n,k+n} R_{k+n}^j = E, \quad j = 1, \dots, s. \quad (21)$$

Take $j = 1$. Now assume that $\limsup_{k \rightarrow \infty} \|Q_k^{1,+}(\lambda)\|^{\frac{1}{k}} \leq 1$. Then for some $C_1 > 0$ and $d, q < d < 1$ we have $\|Q_k^{1,+}(\lambda)\|^{\frac{1}{k}} \leq C_1(1/d)^k$. It means that the norm of the left-hand side of (21) can be majorated by $C_2 q^{k-r} (1/d)^{k-r}$ for some $C_2 > 0$, which tends to zero as $k \rightarrow \infty$. Obviously, this contradicts the identity (21) and therefore we have proved (19) for $j = 1$. By taking $j = 2, \dots, s$ and applying the above arguments we obtain (21) for another values of j .

Also, from (20) follows

$$-R_{k-r:k+s-1}^+ \cdot \begin{bmatrix} \mathbf{0}_{r \times s} & -A_{k-r:k-1,k:k+r-1} \\ A_{k:k+s-1,k-s:k-1} & \mathbf{0}_{s \times r} \end{bmatrix} \cdot Q_{k-s:k+r-1} = I_r.$$

From this identity we get (19) for $Q_k(\lambda)$ similarly as we have done it for $Q_k^+(\lambda)$. □

Finally note that in the scalar case ($A_{i,j} \in \mathbb{C}$) the above results on the operators A were obtained in ([5]) for possibly unbounded operators with $(a_k)_{k \geq 0}$ defined by

$$a_k := \max\{\|A_{k-r:k-1,k:k+r-1}\|, \|A_{k:k+s-1,k-s:k-1}\|\}, \quad k \geq 0,$$

containing a sufficiently dense bounded subsequence. If, instead we consider the matrix case ($A_{i,j} \in \mathbb{C}^{N \times N}$), we can obtain the same results as above, we omit the details.

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